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Real-space renormalisation group for directed systems

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Abstract. A new real-space renormalisation group method is developed to treat systems with directionally dependent critical behaviour. Here two effective lengths $S_{\parallel}(b)$ and $S_{\perp}(b)$ are defined for a given cell of linear size b. The renormalised lattice is constructed from these effective lengths and is deformed from the original lattice. It is the anisotropic rescaling of these effective lengths which give the anisotropic exponents $\nu_{\parallel}$ and $\nu_{\perp}$. This method gives the exact result for the case of fully directed SAWs on a square lattice in the $b \to \infty$ limit. For directed bond and directed site lattice animals, the results obtained here are also in good agreement with the other known values.

1. Introduction

The systems with directionally dependent critical phenomena have been the focus of much study in the past few years. These systems are directed percolation (see e.g. Kinzel 1983), directed lattice animals (see e.g. Nadal *et al* 1983) and directed self-avoiding walks (SAWs) (Redner and Majid 1983). It is well known that the introduction of preferred direction in such systems gives rise two independent correlation lengths $\xi_{\parallel}$ and $\xi_{\perp}$, parallel and perpendicular to the preferred direction respectively. For the case of directed SAWs, the corresponding exponents $\nu_{\parallel}$ and $\nu_{\perp}$ have been obtained exactly with the values $\nu_{\parallel}=1$ and $\nu_{\perp}=\frac{1}{2}$ for all dimensions $d \geqslant 2$ (Redner and Majid 1983, Cardy 1983). For the other systems $\nu_{\parallel}$ and $\nu_{\perp}$ have been calculated by various methods: e.g. Monte Carlo, series expansion and phenomenological renormalisation ($d=2$ only) etc.

Attempts have been made to use the usual real-space renormalisation group (RSRG) method for directed systems (Redner and Yang 1982, Redner 1982). However, only single ν was obtained in these attempts and it was not known how to differentiate $\nu_{\parallel}$ and $\nu_{\perp}$. Phani and Dhar (1982) have pointed out that an anisotropic scaling with two different rescaling factors $b_{\parallel}$ and $b_{\perp}$ of the lattice have to be used in RSRG in order to obtain two independent $\nu_{\parallel}$ and $\nu_{\perp}$, with $b_{\parallel}$ and $b_{\perp}$ related by $b_{\parallel}=b_{\perp}^{\theta}$ with $\theta=\nu_{\parallel}/\nu_{\perp}$. Since only certain ratios of $b_{\parallel}/b_{\perp}$ can be realised with reasonable cell sizes, this method is not very feasible (Herrmann *et al* 1983). To avoid the problem of changing cell shapes, Herrmann *et al* (1983) used the trick of patching square cells together to form an infinite sequence. They were able to obtain the correct value of $\nu_{\parallel}$ in the $b \to \infty$ limit.

In this paper, we present a new RSRG method for directed systems. Instead of choosing an anisotropic cell of linear sizes $b_{\parallel}$ and $b_{\perp}$, we use an isotropic cell of linear size b so that the recursion relation can be obtained by conventional methods. To introduce an anisotropy, we define two effective lengths $S_{\parallel}$ and $S_{\perp}$ for both the original and renormalised lattices. These effective lengths are taken as the basic units for the

measurements of correlation lengths $\xi_\parallel$ and $\xi_\perp$. It is the anisotropic rescaling of $S_\parallel$ and $S_\perp$ which gives the anisotropic exponents $\nu_\parallel$ and $\nu_\perp$. By doing so, even for small b we are able to obtain reasonable values of $\nu_\parallel$ and $\nu_\perp$.

In order to test our method, we first apply it to the simple case of directed SAWs on a square lattice where the exact results are known. Then, we use this method for the cases of directed bond and directed site lattice animals.

2. Directed SAWs

Firstly, for simplicity, we consider the problem of SAWs on a fully directed square lattice. To perform RSRG, we choose a cell of linear size b (see figure 1(a) for $b=3$). Let K be the fugacity of each directed step in the original lattice. As usual, the renormalisated fugacity K' is defined as the sum of all vertically percolating paths which start at the origin and end on the opposite edge of the cell each weighted by K^n, where n is the number of steps in the path. For the undirected SAWs, one does not have to use this corner rule (Redner and Reynolds 1981). But for the directed systems our method does require the origin to be fixed on the preferred axis of the cell. It is easy to write down the following recursion relation

$$K' = K^b + \binom{b}{1} K^{b+1} + \binom{b+1}{2} K^{b+2} + \ldots + \binom{2b-2}{b-1} K^{2b-1}. \tag{1}$$

One can show by induction that the exact critical fugacity $K^* = \frac{1}{2}$ (Redner and Majid 1983) is indeed the non-trivial fixed point of (1) for all $b \geqslant 2$. In fact, this is the only non-trivial fixed point. Now we define the effective lengths $S_\parallel$ and $S_\perp$. For the original lattice, we decompose a directed one step walk into two components. One parallel and the other perpendicular to the preferred direction ***OY*** (figure 1). The effective lengths $S_\parallel(b=1)$ and $S_\perp(b=1)$ are simply the lengths of these two components. If the lattice constant of the original lattice is taken to be unity, then $S_\parallel(1) = S_\perp(1) = 1/\sqrt{2}$. In the renormalised lattice, for every percolating path through the cell, we project the end point vector ***OR*** into two components: one parallel and the other perpendicular to the preferred direction (figure 1(a)). The effective lengths $S_\parallel(b)$ and $S_\perp(b)$ are taken as the lengths of these two components weighted by this configuration and averaging over all percolating configurations and finally evaluated at the fixed point

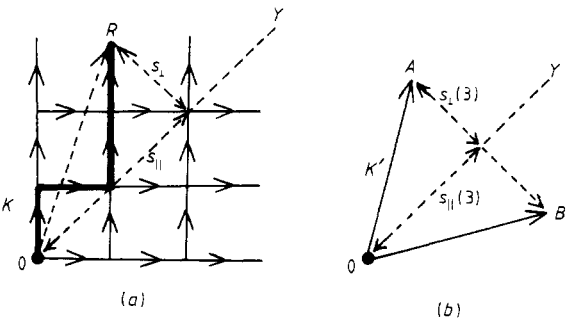

Figure 1. Directed SAWs. (a) A cell of 3×3 is used for transformation. ***OY*** is the preferred direction. Lengths $S_\parallel$ and $S_\perp$ are shown. (b) ***OA*** and ***OB*** are the renormalised directed steps with fugacity K'. These form the basic unit of the renormalised lattice.

K^*. In this simple case, $S_\parallel(b)$ and $S_\perp(b)$ can be written down immediately with the expressions

$$S_\parallel(b)=\frac{1}{\sqrt{2}}\left\{\frac{1}{K'}\left[bK^b+(b+1)\binom{b}{1}K^{b+1}+(b+2)\binom{b+1}{2}K^{b+2}+\ldots\right.\right.$$
$$\left.\left.+(2b-1)\binom{2b-2}{b-1}K^{2b-1}\right]\right\}_{K=K^*} \tag{2}$$

$$S_\perp(b)=\frac{1}{\sqrt{2}}\left\{\frac{1}{K'}\left[bK^b+(b-1)\binom{b}{1}K^{b+1}+(b-2)\binom{b+1}{2}K^{b+2}+\ldots\right.\right.$$
$$\left.\left.+1\binom{2b-2}{b-1}K^{2b-1}\right]\right\}_{K=K^*}. \tag{3}$$

Alternatively, we can also use the second moment average in defining $S_\parallel$ and $S_\perp$. If so, (2) and (3) become

$$S_\parallel^{(2)}(b)=\frac{1}{\sqrt{2}}\left\{\frac{1}{K'}\left[b^2K^b+(b+1)^2\binom{b}{1}K^{b+1}+(b+2)^2\binom{b+1}{2}K^{b+2}+\ldots\right.\right.$$
$$\left.\left.+(2b-1)^2\binom{2b-2}{b-1}K^{2b-1}\right]\right\}^{1/2}_{K=K^*} \tag{4}$$

$$S_\perp^{(2)}(b)=\frac{1}{\sqrt{2}}\left\{\frac{1}{K'}\left[b^2K^b+(b-1)^2\binom{b}{1}K^{b+1}+(b-2)^2\binom{b+1}{2}K^{b+2}+\ldots\right.\right.$$
$$\left.\left.+1^2\binom{2b-2}{b-1}K^{2b-1}\right]\right\}^{1/2}_{K=K^*} \tag{5}$$

where K' in (2)–(5) is given by (1). The reason that the quantities (2)–(5) are evaluated at the fixed point K^* is because we are interested in the region arbitrarily close to the critical point K_c, and K^* will approach K_c as $b\to\infty$. From the above definitions, one can see that in the critical region, $S_\parallel(b)$ and $S_\perp(b)$ are actually the distances made by a step of a renormalised directed walk ***OA*** relative to the preferred direction of the system (figure 1(b)). These renormalised directed steps ***OA*** and ***OB*** form the basic unit of the renormalised lattice. Since the rescaling of $S_\parallel(b)$ and $S_\perp(b)$ are nonlinear in b($S_\parallel(b)$ will become linear in the $b\to\infty$ limit but not for $S_\perp(b)$) the renormalised lattice so a deformed parallelogram lattice is constructed relative to the original lattice (figure 2).

Since the system remains unchanged, except for the change of scales, before and after the transformation, we have following relations

$$\xi_\parallel(K)/\xi_\parallel(K')=S_\parallel(b)/S_\parallel(1) \qquad \text{and} \qquad \xi_\perp(K)/\xi_\perp(K')=S_\perp(b)/S_\perp(1). \tag{6}$$

In the critical region, $\xi_\parallel(K)$ and $\xi_\perp(K)$ behave like

$$\xi_\parallel(K)\sim(K_c-K)^{-\nu_\parallel} \qquad \text{and} \qquad \xi_\perp(K)\sim(K_c-K)^{-\nu_\perp}. \tag{7}$$

The correlation length exponents $\nu_\parallel$ and $\nu_\perp$ can be obtained from (1), (6) and (7) by the well known procedure leading to the relations

$$\nu_\parallel(b,1)=\frac{\ln[S_\parallel(b)/S_\parallel(1)]}{\ln\lambda(b,1)} \tag{8}$$

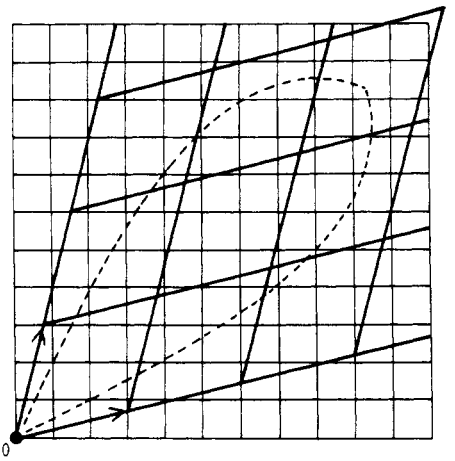

Figure 2. The renormalised lattice (bold lines) is deformed relative to the original lattice. The broken curve denotes the range of correlations in the system which remain unchanged before and after the transformation.

and

$$\nu_\perp(b,1)=\frac{\ln[S_\perp(b)/S_\perp(1)]}{\ln\lambda(b,1)} \tag{9}$$

where $\lambda(b,1)$ is the eigenvalue of the recursion relation at the fixed point K^*. Similarly, one can also perform the cell-to-cell transformation.

In the usual RSRG for the undirected systems, one always projects the end vector of each percolating path onto the vertical (or horizontal) axis. This leads to a constant scaling factor b of the usual RSRG. So, our method described here is actually the generalisation of the conventional RSRG to the directed systems.

Now we calculate $\nu_\parallel$ and $\nu_\perp$ explicitly. From (1) the eigenvalue $\lambda(b,1)$ has the expression

$$\lambda(b,1)\equiv\frac{dK'}{dK|_{K^*}}=bK^{b-1}+(b+1)\binom{b}{1}K^b+(b+2)\binom{b+1}{2}K^{b+1}+\ldots+(2b-1)\times\binom{2b-2}{b-1}K^{2b-2}. \tag{10}$$

It is easy to see from (2) and (10) that $\lambda(b,1)=\sqrt{2}S_\parallel(b)$. From (8) we find $\nu_\parallel=1$ for all $b\geq 2$. The agreement with the exact result even for small b is surprising. For $K^*=\frac{1}{2}$, $S_\perp(b)$ and $\lambda(b,1)$ of (3) and (10) can be summed up. With some manipulations, we find

$$S_\perp(b)=\frac{1}{\sqrt{2}}\frac{(2b-1)!}{[(b-1)!]^2}\left(\frac{1}{2}\right)^{2b-2} \tag{11}$$

$$\lambda(b,1)=2b\left[1-\frac{(2b)!}{(b!)^2}\left(\frac{1}{2}\right)^{2b}\right]. \tag{12}$$

Using (11), (12) and (9) we get $\nu_\perp(2,1)=0.4425$, $\nu_\perp(3,1)=0.4436$ and cell-to-cell result $\nu_\perp(3,2)=0.4456$. These small cell results are also in reasonably good agreement with the exact value $\nu_\perp=0.5$. When b is large, using Stirling's formula, (11) and (12) become

$$S_\perp(b)\sim(2b/\pi)^{1/2} \tag{13}$$

$$\lambda(b,1)\sim 2b(1-1/\sqrt{\pi b}). \tag{14}$$

From (9), we find

$$\nu_\perp(b)\sim\tfrac{1}{2}+O(1/\ln b). \tag{15}$$

So, the exact value of $\nu_\perp$ is obtained in the $b\to\infty$ limit. However, the convergence is rather slow in this case. If we use the second moment difinitions of $S_\parallel^{(2)}(b)$ and $S_\perp^{(2)}(b)$ in (4) and (5), exact results are also obtained in the $b\to\infty$ limit.

In order to show that our method is not limited to the particular square cells chosen in figure 1(a), we choose another kind of cell as shown in figure 3. Since the preferred axis is ***OY***, the origin must be fixed at the point 0. All directed SAWs starting at 0 traversing the cell vertically are considered as percolated. For such a cell, obviously, we have $S_\parallel(b)=b$. For simplicity, only cells with even b are used to perform the cell-to-cell transformation. It is not difficult to show that $\nu_\parallel$ is again equal to 1 for all even b's. In figure 4 the results for K^* and $\nu_\perp$ are plotted against $\ln((b+4)/b)$. The exact results are again obtained in the $b\to\infty$ limit. However, in the following calculations we will only use the square cells (figure 1(a)) because it is much simpler.

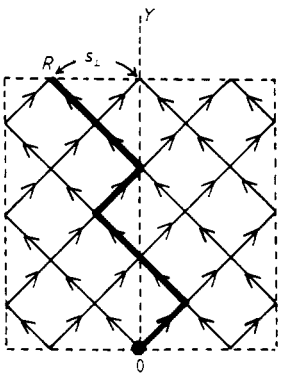

Figure 3. Directed SAWs. A different kind of cell is used for transformations. Here we show the $b=6$ cell.

As we have mentioned before, our method is the generalisation of the conventional RSRG. If the 'directed' restriction is lifted in our previous calculations, we would expect that both $\nu_\parallel$ and $\nu_\perp$ will approach the ν value of the undirected SAWs. This point has been checked by performing undirected cell-to-cell transformations. The values of K^*, $\nu_\parallel$ and $\nu_\perp$ are plotted against $\ln(b/b')$ in figure 5. We find that the values $K_c=0.379$ and $\nu=0.75$ obtained by Derrida (1981) for undirected SAWs are indeed recovered in the $b\to\infty$ limit. So, our method is not limited to the directed systems. It provides another way to treat undirected systems using the RSRG method.

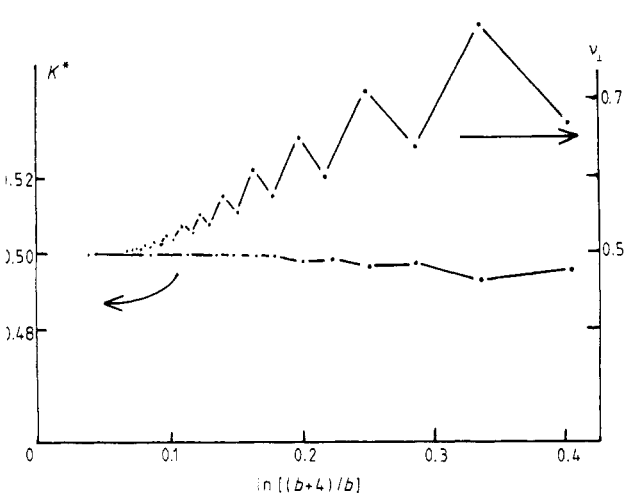

Figure 4. Directed SAWs. Using the kind of cell shown in figure 3, cell-to-cell transformation results are plotted against $\ln[(b+4)/b]$. Right axis for $\nu_\perp$. Left axis for K^*.

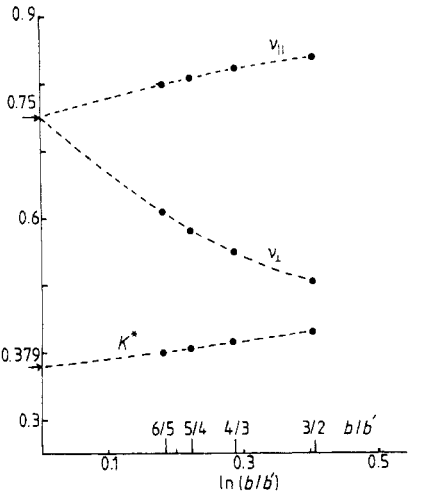

Figure 5. Undirected SAWs. The values of K^*, $\nu_\parallel$ and $\nu_\perp$ obtained from cell-to-cell transformations are plotted against $\ln(b/b')$.

3. Directed lattice animals

Now we apply our method to the more interesting case of directed lattice animals on a square lattice where the critical behaviour is non-classical. For the fully directed bond lattice animals, a cell of linear size $b = 3$ is shown in figure 6(a). A single origin is again fixed at point 0. All the percolating clusters which traverse the cell vertically are renormalised to a directed bond ***OA*** (figure 6(b)) with fugacity K'. The recursion relations for $b \leq 4$ have been given by Redner and Yang (1982). The effective lengths $S_\parallel(b)$ and $S_\perp(b)$ are defined in the following ways. For every percolating cluster, we project the upper-left part of the cluster onto two axes: one the preferred axis and the other perpendicular to it. These projections define $S_\parallel$ and $S_\perp$ of this configuration (figure 6(a)). Weighted by this configuration and averaging over all percolating configurations we obtain $S_\parallel(b)$ and $S_\perp(b)$. The basic units ***OA*** and ***OB*** (figure 6(b))

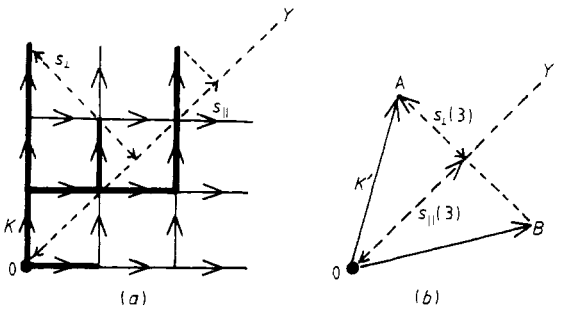

Figure 6. Directed bond lattice animals. (a) A cell of 3×3 is shown. ***OY*** is the preferred direction. Lengths $S_\parallel$ and $S_\perp$ are also shown. (b) ***OA*** and ***OB*** are the renormalised directed bonds with fugacity K'. These form the basic unit of the renormalised lattice.

of the renormalised lattice are then obtained. From (8) and (9) one can calculate $\omega_{\parallel}$ and $\omega_{\perp}$. The results are given in table 1. These results are in reasonably good agreement with values $K_c = 0.2851$, $\nu_{\parallel} = 0.8180$ and $\nu_{\perp} = 0.498$ found by Nadal *et al* (1982). Since the values of $\nu_{\parallel}$ in table 1 show a small fluctuation, we think a larger cell calculation is required to determine $\nu_{\parallel}$ and $\nu_{\perp}$ more accurately.

Table 1. Directed bond lattice animals. Results for K^*, $\nu_{\parallel}$ and $\nu_{\perp}$ for the rescaling of a cell of linear size b to a cell of linear size b' (in renormalised units).

	b \ b'	1	2	3
K^*	2	0.3306		
$\nu_{\parallel}$		0.7945		
$\nu_{\perp}$		0.4879		
K^*	3	0.3157	0.3048	
$\nu_{\parallel}$		0.7976	0.8013	
$\nu_{\perp}$		0.4993	0.5132	
K^*	4	0.3080	0.3003	0.2966
$\nu_{\parallel}$		0.7972	0.7984	0.7941
$\nu_{\perp}$		0.5041	0.5142	0.5146

Finally, we consider the case of fully directed site lattice animals on a square lattice. As before, a cell of b^2 sites is chosen with a single fixed origin at the lower-left corner of the cell (figure 7). Here we use both the R_0 and R_1 rules (Reynolds *et al* 1980) to define when the cell percolates. To obtain $S_{\parallel}$ and $S_{\perp}$, unlike the bond case, the whole percolating cluster is projected onto two axes: one parallel and the other perpendicular to the preferred direction. The second moment definitions $S_{\parallel}^{(2)}(b)$ and $S_{\perp}^{(2)}(b)$ are similarly defined. Since $S_{\parallel}(b=1)$ and $S_{\perp}(b=1)$ are equal to zero, cell-to-cell transformations must be used here. The renormalised cell contains $(b')^2$ sites with $b > b' \geq 2$. The results are given in tables 2(*a*) and 2(*b*) for the R_0 and R_1 rules respectively, where $\nu_{\parallel}^{(2)}$ and $\nu_{\perp}^{(2)}$ are the results obtained from the second moments, $S_{\parallel}^{(2)}$ and $S_{\perp}^{(2)}$, calculations. Since the results obtained from the second moments are very close to the first moment results, we only give those results for the R_0 rule for comparison. Table 2 also shows that the differences between R_0 and R_1 rules are

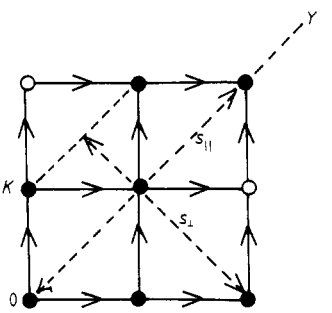

Figure 7. Directed site lattice animals. A cell of 3×3 sites ($b = 3$) is shown. ***OY*** is the preferred direction. Lengths $S_{\parallel}$ and $S_{\perp}$ are also shown.

Table 2. Directed site lattice animals. Results for K^*, $\nu_\parallel$ and $\nu_\perp$ for the rescaling of a cell of b^2 sites to a cell of $(b')^2$ sites, (a) using R_0 rule, (b) using R_1 rule.

(a)

	b \ b'	2	3	4	5
K^*	3	0.3528			
	4	0.3500	0.3472		
	5	0.3475	0.3453	0.3436	
	6	0.3457	0.3439	0.3425	0.3414
$\nu_\parallel$	3	1.3203			
	4	1.2268	1.0942		
	5	1.1736	1.0559	1.0061	
	6	1.1384	1.0305	0.9849	0.9588
$\nu_\parallel^{(2)}$	3	1.2780			
	4	1.1930	1.0722		
	5	1.1443	1.0369	0.9910	
	6	1.1121	1.0135	0.9714	0.9472
$\nu_\perp$	3	1.1094			
	4	1.0141	0.8786		
	5	0.9588	0.8375	0.7839	
	6	0.9217	0.8098	0.7602	0.7310
$\nu_\perp^{(2)}$	3	1.1068			
	4	1.0091	0.8701		
	5	0.9536	0.8302	0.7782	
	6	0.9168	0.8034	0.7553	0.7269

(b)

	b \ b'	2	3	4
K^*	3	0.3670		
	4	0.3598	0.3535	
	5	0.3551	0.3502	0.3472
$\nu_\parallel$	3	1.2934		
	4	1.2068	1.0836	
	5	1.1570	1.0472	1.0000
$\nu_\perp$	3	1.0837		
	4	0.9976	0.8741	
	5	0.9466	0.8354	0.7848

rather small. In figure 8 critical quantities K^*, $\nu_\parallel$ and $\nu_\perp$ are plotted against $\ln(b/b')$ for the first moment results with R_0 rule. The results of Nadal *et al* (1982) with $K_c = \frac{1}{3}$, $\nu_{11} = \frac{9}{11}$ and $\nu_\perp = \frac{1}{2}$ are indeed approached when $b \to \infty$.

4. Concluding remarks

We have developed a new RSRG method to treat systems with directionally dependent critical behaviour. For a given cell of linear size b, two effective lengths $S_\parallel(b)$ and

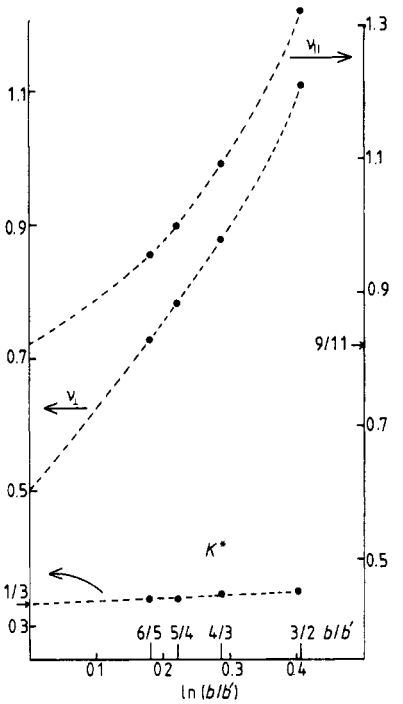

Figure 8. Directed site lattice animals. Cell-to-cell transformation results for K^*, $\nu_\parallel$ and $\nu_\perp$ are plotted against $\ln(b/b')$. Left axis for K^* and $\nu_\perp$. Right axis for $\nu_\parallel$.

$S_\perp(b)$ are defined. A deformed renormalised lattice is constructed from these effective lengths. These effective lengths form the basic units for the measurements of correlation lengths $\xi_\parallel(K')$ and $\xi_\perp(K')$ in the renormalised lattice. It is the anisotropic rescaling of these effective lengths $S_\parallel(b)$ and $S_\perp(b)$ which give the anisotropic exponents $\nu_\parallel$ and $\nu_\perp$.

Our method is the generalisation of the usual RSRG method to directed systems. when 'directed' restriction is lifted, both $\nu_\parallel$ and $\nu_\perp$ are expected to approach the isotropic ν of the corresponding undirected systems when b becomes large. So, it also provides another method to study undirected systems.

When we apply this method to directed SAWS, exact results are obtained in the $b \to \infty$ limit. For the cases of directed bond and directed site lattice animals, our results are also in good agreement with the phenomenological renormalisation calculations.

We have also used this method on three-dimensional directed systems where the phenomenological renormalisation is not applicable. Since only a $2 \times 2 \times 2$ cell is used, the results are not very good. A large-cell Monte Carlo renormalisation method is required to give reliable results.

When this method is applied to directed percolation systems in two dimensions, the single-root method can not be used because it always overestimates the critical percolation p_c. In this case the cell shown in figure 3 with the edge-to-edge rule has to be used. Again large-cell Monte Carlo renormalisation calculations are required to give reliable results. Efforts have been made in this direction.

Acknowledgments

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